PUTNAM PRACTICE SET 5

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Problem 1. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We consider a function $f : \mathbb{N} \longrightarrow \mathbb{N}_0$ satisfying the following properties:

(a) for any $m, n \in \mathbb{N}$, we have that $f(m+n) - f(m) - f(n) \in \{0, 1\}$
(b) $f(2) = 0$;
(c) $f(3) > 0$; and
(d) $f(9999) = 3333$.

Compute $f(2019)$.

Problem 2. Find all real numbers a for which the equation

$$16x^4 - ax^3 + (2a+17)x^2 - ax + 16 = 0$$

has 4 distinct real roots which form a geometric progression.

Problem 3. Let $P(x)$ be a monic polynomial of degree 3 with integer coefficients. If one of its roots equals the product of the other two roots, then prove that there exists an integer m such that

$$2P(-1) = m \cdot (P(1) + P(-1) - 2 - 2P(0)).$$

Problem 4. Let $m, n \in \mathbb{N}$. In a box there are m white balls and n black balls. We extract randomly two balls from the box; if the two balls have different colors, then we put back in the box a white ball, while if the two balls have the same color, then we put back in the box a black ball. We repeat this procedure until there is left in the box only one single ball. What is the probability that this last ball is white?